Low rank and sparse dynamical maps and repeated entries in the process matrix

Vinayak Jagadish
Quantum Research Group, School of Chemistry and Physics,
University of KwaZulu-Natal, Durban, KwaZulu-Natal, 4001, South Africa and
National Institute for Theoretical Physics (NITheP), KwaZulu-Natal, 4001, South Africa

Anil Shaji
School of Physics, IISER TVM, CET Campus,
Thiruvananthapuram, Kerala, India 695016

We show that the process matrix in the basis of tensor products of Pauli operators or SU(N) generators representing low rank and sparse dynamical maps will have only a few distinct entries which goes as $O(r^2)$ (r is the rank).

Understanding open quantum dynamics [1] better, developing strategies to control it [2, 3] and application to new technologies has gained significant interest. Quantum process tomography is the usual technique employed for tracking the unknown dynamics of quantum systems [4–7]. The prime difficulty in performing tomography tasks for large systems is the growth of resources needed for reconstructing the processes. For an n qubit system $N = 16^n - 4^n$ independent measurements are required for process tomography when done in the conventional way [8]. In many situations, physical considerations regarding the process that is to be reconstructed and the structure of the problem at hand allows one to reduce the resources required to do process tomography. Indeed there has been considerable progress in reducing the number of independent measurements required for characterizing noisy processes [9–13]. In particular, extensions of the compressed sensing and matrix completion techniques [14, 15] that have been used effectively for quantum state tomography when the states are described by sparse density matrices [16, 17] to process tomography [18] provide substantial reduction in the resources required under certain conditions.

I. REPRESENTATIONS OF QUANTUM PROCESSES AND PROPERTIES

The finite time open dynamics of a quantum system is described by a quantum process (or operation) represented by a map, $\mathcal{E} : \rho \to \mathcal{E}(\rho)$. Since quantum operations take density matrices to density matrices, the map, in general, has to be convex-linear, trace preserving and completely positive on all the states in its domain. The Dynamical Matrix is a matrix representation of the map [19] that acts on a density matrix as $\rho_{r's'} \to \mathcal{E}(\rho)_{rs} = \mathfrak{B}_{rr',ss'}\, \rho_{r's'}$. If ρ is d dimensional, then $\mathfrak{B}$ is a d^2 dimensional Hermitian matrix with eigenvalues λ_n and eigenvectors $|e_n\rangle$.

The canonical Kraus operators [20] are the matricized [21] versions of the eigenvectors of $\mathfrak{B}$ such that $K_n = \sqrt{\lambda_n}\,\mathrm{mat}|e_n\rangle$. The mat operation stacks the elements of a column matrix row by row with rows of length d to generate a $d \times d$ square matrix. The number of canonical Kraus operators

is equal to the rank r of the dynamical matrix. The map in the Kraus (operator-sum) form is:

$$\mathcal{E}(\rho) = \sum_{n=1}^{r} K_n \rho K_n^\dagger \quad \text{where} \quad \sum_{n=1}^{r} K_n^\dagger K_n = \mathbb{1} \tag{1}$$

The Kraus representation is not unique, since each K_n can be multiplied on the left by a unitary matrix without violating the only constraint on the Kraus matrices given by the second equation in (1). Each of the Kraus operators for a map $\mathcal{E}$ can be expanded in a suitable operator basis $\{A_i\}$ as $K_n = \sum_i \alpha_i^{(n)} A_i$, with $\alpha_i^{(n)} \in \mathbb{C}$. The operator basis can be chosen to be orthonormal ($\mathrm{tr}[A_j^\dagger A_k] = \delta_{jk}$) for convenience. Equation (1) can then be written as

$$\mathcal{E}(\rho) = \sum_{ij} \chi_{ij} A_i \rho A_j^\dagger, \quad \text{where} \quad \chi_{ij} = \sum_n \alpha_i^{(n)} \alpha_j^{(n)*} \tag{2}$$

So for a given basis set $\{A_i\}$ the matrix χ completely characterizes $\mathcal{E}$. The χ matrix is Hermitian and different Kraus representations of the same process $\mathcal{E}$ have the same χ matrix. From here on, we call the χ matrix, the process matrix.

In a typical process tomography experiment on a system made of n qubits, the $\{A_i\}$'s are usually taken to be the n-fold tensor products of Pauli operators. To characterize an unknown operation, one prepares a complete set of linearly independent input states, subject them to the quantum operation $\mathcal{E}$ and determine the output states corresponding to each input using quantum state tomography. The details of performing a Standard Quantum Process Tomography (SQPT) can be found in [7]. We assume that the input states are initialized such that they are uncorrelated with the environment and hence the map that is reconstructed is completely positive.

II. RESULT AND DISCUSSION

We focus on dynamical maps $\mathfrak{B}$ which are sparse and low rank. Our main result is purely based on the properties of the process matrix. The experimental accessibility of the χ matrix is the main motivation for studying its properties in detail for the low rank quantum operations considered here. Since Pauli measurements are relatively easy to perform in real quantum process tomography experiments, our results can be readily tested as well. For such processes we show that number of distinct entries in the χ matrix goes as $O(r^2)$.

The χ matrix can be written as an outer product as follows

$$\chi = \sum_{i=1}^{r} \mathfrak{L}_i \widetilde{\mathfrak{L}}_i \tag{3}$$

where

$$\mathfrak{L}_i = \left[K_i^{(1)}, K_i^{(2)}, \ldots, K_i^{(d)}\right]^T, \qquad \text{with} \qquad K_i^{(d)} = \mathrm{tr}(K_i \lambda_d),$$

and

$$\widetilde{\mathfrak{L}}_i = \left[k_i^{(1)}, k_i^{(2)}, \ldots, k_1^{(d)}\right], \qquad \text{with} \quad k_i^{(d)} = \mathrm{tr}(K_i^T \lambda_d).$$

Here $\{\lambda_i\}$ denotes the convenient operator basis used for defining the χ matrix and the superscript T denotes the transpose operation. For the discussion that follows we take the basis to be made of the n-fold tensor products of Pauli matrices.

The product $\mathfrak{L}_i\widetilde{\mathfrak{L}}_i$ is Hermitian and so is χ. The elements of $\mathfrak{L}_i$ are conjugates of $\widetilde{\mathfrak{L}}_i$. Let us assume that the Kraus matrices, K_i, each have only a maximum of r non-zero entries. For an s-sparse matrix, it's rank satisfies the bound $O(1) \leq r \leq s$ and also a tighter bound $O(1) \leq r \leq \min(s, d)$ for a d dimensional matrix. [22]. The $\mathfrak{L}_i$, which is the the matrix constructed out of the trace with λ_d's of the Kraus matrices, will have $O(r)$ distinct entries. Consider the trace of an arbitrary 2×2 matrix with the Pauli matrix σ_1

$$\mathrm{Tr}\left[\begin{pmatrix} a & b \\ x & y \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = b + x$$

Only two elements b and x (the circled ones) contribute to the trace. Since the trace operation is with Pauli matrices or their tensor products, it is evident that only d elements in the case of a $d \times d$ matrix will contribute to the trace, since the Pauli matrices or their tensor products have only d non zero entries. Consider an r-sparse Kraus matrix which means that it has only r non-zero entries.

$$\begin{aligned} \text{The probability of an entry being non- zero} &= \frac{r}{d^2} \\ \text{The probability that only one element contributes to trace} & \\ &= {}^dC_1 \left(1 - \frac{r}{d^2}\right)^{d-1} \frac{r}{d^2} \end{aligned}$$

The probability that only one element from each of the r Kraus operators contributing to the trace is high. Remember that $r << d^2$ which validates the high probability argument. If they are located in such a way that only one element contributes towards its trace with a particular λ_i, then $4r$ different possibilities alone arise, since the trace with Pauli tensors produces ± 1 or $\pm i$ alone. This indicates that if more elements contribute to the trace with a particular λ_i, the number of distinct entries scales linearly with r.

From Eq. (3), it is therefore clear that the number of distinct entries in the χ matrix is additive w.r.t to the Kraus matrices and since they are r in number, the number goes as $O(r^2)$. It can be understood from this construction that if different Kraus matrices have the same matrix positions of non-zero entries the *number* of distinct entries will not increase.

Consider the case where only a single non zero element is present in the Kraus operators, but their positions are different.

$$M_1 = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}$$

$$\chi_1 = \begin{pmatrix} a_1^2 & 0 & 0 & a_1^2 \\ 0 & a_2^2 & -ia_2^2 & 0 \\ 0 & ia_2^2 & a_2^2 & 0 \\ a_1^2 & 0 & 0 & a_1^2 \end{pmatrix}$$

The χ matrix has two non zero entries.
Consider a second case as follows

$$L_1 = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix}$$

$$\chi_2 = \begin{pmatrix} a_1^2 + a_2^2 & 0 & 0 & a_1^2 - a_2^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1^2 - a_2^2 & 0 & 0 & a_1^2 + a_2^2 \end{pmatrix}$$

The χ matrix has two non zero entries.

It is worth noting that the χ matrix in the basis of SU(N) generators will also have the same number of distinct elements due to the fact that the trace of Kraus matrices with generators of SU(N) Lie algebra also produces ± 1 or $\pm i$ alone.This means that the χ matrix written in the basis of SU(N) generators will also have only $O(r^2)$ distinct entries.

III. NUMERICAL SIMULATION

The histograms shown below clearly indicates that the number of distinct entries in the χ matrix are peaked at low numbers which is less than r^2. The histograms were made by constructing χ matrices from sparse Kraus matrices, stemming from sparse and low rank dynamical matrices and counting the number of distinct entries in absolute values. For rank 3, we created three 8×8 matrices each having only 3 entries by allocating the non zero entries in any three random positions of each of the three matrices such that Eq. 1 is satisfied and the number of distinct elements in absolute values was counted for the χ matrix developed in the basis of $\sigma_i \otimes \sigma_j \otimes \sigma_k$. This was repeated for 100,000 realizations by choosing all possible permutations of the location of the non zero entries which satisfied Eq. 1. A similar process was repeated for rank 2 case as well where we have only two Kraus matrices to start off with.

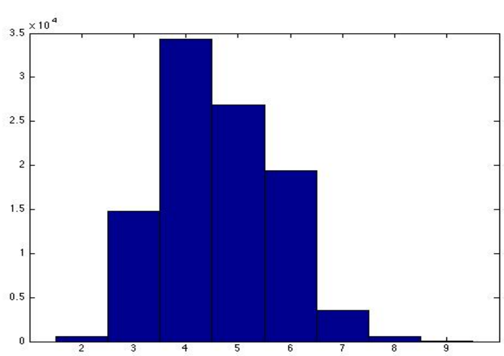

FIG. 1. (Color online) Histogram of 100,000 realizations showing the number of distinct matrix elements (in absolute values) of a 64 dimensional χ matrix corresponding to a dynamical matrix of rank 2.

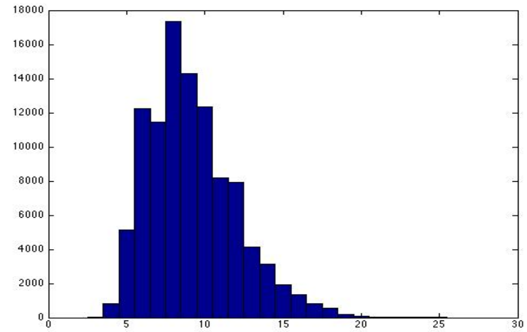

FIG. 2. (Color online) Histogram of 100,000 realizations showing the number of distinct matrix elements (in absolute values) of a 64 dimensional χ matrix corresponding to a dynamical matrix of rank 3.

IV. CONCLUSION

We have shown that when the underlying process is represented by a sparse dynamical matrix, then in the χ matrix form, a lot of entries are repeated. Thus a χ matrix reconstructed from process tomography data with many repeated entries is another indicator for the underlying process to be low rank and sparse.

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